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Optimal Existence Criteria for Symmetric Positive Solutions of Third-order Singular Boundary Value Problems*

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Abstract: In this paper, based on the Leggett-Williams fixed point theorem together with constructing a special cone, some new optimal existence results of triple symmetric positive solutions of third-order singular boundary value problems are obtained. Finally, an example demonstrates the validity of the main results.

Keywords: symmetric positive solution; existence; boundary value problem

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1 Introduction

We consider the existence of symmetric positive solutions for the following third-order singular boundary value problem (BVP)

$$\begin{cases} z''' + f(t, z(t)) = 0, & t \in (0, 1), \\ z(0) = z'(0) = z(1) = 0, \end{cases} \quad (1)$$

where $f : (0, 1) \times [0, +\infty) \rightarrow [0, +\infty)$ satisfies Caratheodary conditions and $f(t, z(t))$ may be singular at $t = 0$ and/or $t = 1$.

Boundary value problems arise in a variety of different areas of applied mathematics and physics (see [1,2], and the references therein). Recently, many authors have studied the existence of positive solutions for singular boundary value problems (see [3-5] and the references therein), and there are many papers focused on third-order boundary value problems. However, successful results about the existence of symmetric positive solutions to the BVP (1) are few.

Inspired and motivated by the results in [3] and [5], this paper try to establish some optimal existence criteria for the existence of triple symmetric solutions to the BVP (1) by applying a fixed point theorem.

2 Preliminaries

We consider the BVP (1) in a Banach space $E = C[0, 1]$ equipped with the norm

$$\|z\| = \max_{0 \leq t \leq 1} |z(t)|, \quad I = [0, 1]$$

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for any $z \in E$. Let K be a cone of E , and u be a nonnegative continuous concave function on I , and m, n be constants, $0 < m < n$. Denote

$$K_r = \{z \in K \mid \|z\| < r\}, \quad \overline{K}_r = \{z \in K \mid \|z\| \leq r\},$$

$$K(u, m, n) = \{z \in K \mid m \leq u(z), \|z\| \leq n\}.$$

Lemma 2.1^[6] Let $A : K \rightarrow K$ be a completely continuous operator, u be a nonnegative continuous concave function on K , and satisfies $u(z) \leq \|z\|$, for all $z \in \overline{K}_r$. In addition, assume that there exist $0 < d < m < n \leq r$ satisfy the following conditions:

- (i) $\{z \in K(u, m, n) \mid u(z) > m\} \neq \emptyset$, and $u(Az) > m$ for $z \in K(u, m, n)$;
- (ii) $\|Az\| \leq d$ for $z \in \overline{K}_d$;
- (iii) $u(Az) > m$ for $z \in K(u, m, r)$ and $\|Az\| > n$;

then A has at least three fixed points z_1, z_2, z_3 on \overline{K}_r satisfies $\|z_1\| < d$, $m < u(z_2)$, and $\|z_3\| > d$ for $u(z_3) < m$.

Let $G(t, s)$ be the Green's function of the homogeneous linear problem of the BVP (1), that is

$$G(t, s) = \begin{cases} \frac{1}{2}s^2(1-t)^2 + s(1-t)(t-s), & 0 \leq s \leq t \leq 1, \\ \frac{1}{2}t^2(1-s)^2, & 0 \leq t \leq s \leq 1. \end{cases}$$

It is easy to prove the following properties of the Green's function:

$$(I) \quad G(\mu(s), s) = \frac{s(1-s)^2}{2(2-s)}, \text{ where } G(\mu(s), s) = \max_{0 \leq t \leq 1} G(t, s) \text{ and } \mu(s) = \frac{1}{2-s}, s \in [0, 1].$$

$$(II) \quad a(t)G(\mu(s), s) \leq G(t, s) \leq G(\mu(s), s), (t, s) \in [0, 1] \times [0, 1], \text{ where } a(t) = t^2(1-t).$$

$$(III) \quad \max_{0 \leq t \leq 1} \int_0^1 G(t, s) ds = \frac{1}{3}(3 \ln 2 - 2).$$

$$(IV) \quad \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G(t, s) ds = \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{4}, s\right) ds = \frac{13}{3072}.$$

$$(V) \quad \min_{0 \leq c \leq 1} \frac{G(\frac{1}{4}, c)}{G(\frac{1}{2}, c)} = \frac{1}{4}.$$

3 Main results

Theorem 3.1 Suppose the following conditions hold:

(H₁) $f \in C((0, 1) \times [0, +\infty), [0, +\infty))$, $f(t, z) \leq g(t)h(z)$, $g \in C((0, 1), [0, +\infty))$, $h \in C([0, +\infty), [0, +\infty))$;

(H₂)

$$0 < \int_0^1 G(\mu(s), s)g(s)ds < +\infty;$$

(H₃) There exist $0 < d < m < \frac{r}{2}$ such that

$$1) \quad h(z) \leq d \left[\int_0^1 G(\mu(s), s)g(s)ds \right]^{-1}, \text{ for } 0 \leq z \leq d;$$

$$2) \quad f(t, z) > m \left[\int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{4}, s\right) ds \right]^{-1} = \frac{3072}{13} m, \text{ for } m \leq z \leq 2m;$$

$$3) \quad h(z) < r \left[\int_0^1 G(\mu(s), s) g(s) ds \right]^{-1}, \text{ for } 0 \leq z \leq r;$$

then the BVP (1) has triple symmetric positive solutions z_1, z_2, z_3 satisfy $\|z_1\| < d$, $m < u(z_2)$, and $\|z_3\| > d$ for $u(z_3) < m$.

Proof Denote $C^+[0, 1] = \{z \in C[0, 1] \mid z(t) \geq 0, t \in [0, 1]\}$, and

$$K = \{z \in C^+[0, 1] \mid z \text{ is concave } z(t) = z(1-t), t \in [0, 1]\}.$$

Let

$$u(z) = \min_{t \in [\frac{1}{4}, \frac{3}{4}]} z(t) \quad \text{for all } z \in K.$$

Then

$$u(z) = z\left(\frac{1}{4}\right) \leq z\left(\frac{1}{2}\right) = \|z\|.$$

It is well known that $z \in C[0, 1] \cap C^2(0, 1)$ is a positive solution of the BVP (1) if and only if $z \in C[0, 1]$ is a positive solution of the equation

$$z(t) = \int_0^1 G(t, s) f(s, z(s)) ds.$$

Define operator $A : K \rightarrow K$ by

$$Az(t) = \int_0^1 G(t, s) f(s, z(s)) ds.$$

Obviously, $Az(t) \geq 0$ for all $z \in K$, and it is easy to see $(Az)'(t) \geq 0$, for $t \in (0, \frac{1}{2}]$, $(Az)'(t) \leq 0$, for $t \in [\frac{1}{2}, 1)$, and $Az(t) = Az(1-t)$ for $0 < t < 1$. Consequently $Az \in K$, that is $A : K \rightarrow K$. By Arzela-Ascoli theorem, we can prove $A : K \rightarrow K$ is completely continuous, see [3].

From (H_1) and 3) in (H_3) , for any $z \in \overline{K}_r$, we know that

$$\begin{aligned} \|Az\| &\leq \int_0^1 G(\mu(s), s) g(s) h(z(s)) ds \\ &\leq \int_0^1 G(\mu(s), s) g(s) ds \cdot r \left[\int_0^1 G(\mu(s), s) g(s) ds \right]^{-1} = r. \end{aligned}$$

Hence $A(\overline{K}_r) \subset \overline{K}_r$.

Choose $z(t) = 2m$, $0 < t < 1$, then $z(t) \in K(u, m, 2m)$, and $u(z) = u(2m) > m$. Thus $\{z \in K(u, m, 2m) \mid u(z) > a\} \neq \emptyset$. On the other hand, for $z \in K(u, m, 2m)$, we know that $u(z) = z(\frac{1}{4}) \geq m$. So $m \leq z(s) \leq 2m$, $\frac{1}{4} \leq s \leq \frac{3}{4}$. Thus for any $z \in K(u, m, 2m)$, from 2) in (H_3) , we get

$$\begin{aligned} u(Az) &= \min_{t \in [\frac{1}{4}, \frac{3}{4}]} Az(t) = \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \int_0^1 G(t, s) f(s, z(s)) ds \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{4}, s\right) f(s, z(s)) ds > \frac{3072}{13} m \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{4}, s\right) ds = m. \end{aligned}$$

Then condition (i) of Lemma 2.1 hold.

Next, condition (ii) of Lemma 2.1 follows immediately from (H_1) and 1) in (H_3) . For any $z \in \bar{K}_d$, we have

$$\begin{aligned}\|Az\| &\leq \max_{t \in [0,1]} \int_0^1 G(t,s)g(s)h(z(s))ds \\ &\leq \int_0^1 G(\mu(s),s)g(s)ds \cdot d \left[\int_0^1 G(\mu(s),s)g(s)ds \right]^{-1} = d.\end{aligned}$$

Hence $A : \bar{K}_d \longrightarrow \bar{K}_d$.

Finally, we prove $u(Az) > m$ for $z \in K(u, m, r)$ and $\|Az\| > 4m$.

In fact, from 2) in (H_3) , for $z \in K(u, m, r)$ and $\|Az\| > 4m$, we know that

$$\begin{aligned}u(Az) &= Az\left(\frac{1}{4}\right) = \int_0^1 G\left(\frac{1}{4}, s\right)f(s, z(s))ds = \int_0^1 \frac{G(\frac{1}{4}, s)}{G(\frac{1}{2}, s)}G\left(\frac{1}{2}, s\right)f(s, z(s))ds \\ &\geq \min_{0 \leq c \leq 1} \frac{G(\frac{1}{4}, c)}{G(\frac{1}{2}, c)} \int_0^1 G\left(\frac{1}{2}, s\right)f(s, z(s))ds = \frac{1}{4}Az\left(\frac{1}{2}\right) = \frac{1}{4}\|Az\| > m.\end{aligned}$$

Therefore, the condition (iii) of Lemma 2.1 hold.

By Lemma 2.1, we conclude that A has triple symmetric fix points. Consequently, the BVP (1) has triple symmetric solutions z_1, z_2, z_3 satisfy $\|z_1\| < d$, $m < u(z_2)$ and $\|z_3\| > d$ for $u(z_3) < m$.

Remark 1 The results of Theorem 3.1 is new. Theorem 3.1 also holds when nonlinearity $f(t, z(t))$ is nonsingular at $t = 0$ and $t = 1$.

Theorem 3.2 Let $f(t, z(t)) = g(t)h(z(t))$ and the following conditions hold:

(H'_1) $h \in C([0, +\infty), [0, +\infty))$, $g \in C((0, 1), [0, +\infty))$;

(H'_2) $0 < \int_0^1 G(\mu(s), s)g(s)ds < +\infty$, $\int_{\frac{1}{4}}^{\frac{3}{4}} g(s)ds > 0$;

(H'_3) there exist $0 < d < m < \frac{r}{2}$ such that

$$1) \quad h(z) < d \left[\max_{t \in [0,1]} \int_0^1 G(t,s)g(s)ds \right]^{-1}, \text{ for } 0 \leq z \leq d;$$

$$2) \quad h(z) > m \left[\int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{4}, s\right)g(s)ds \right]^{-1}, \text{ for } m \leq z \leq 2m;$$

$$3) \quad h(z) < r \left[\max_{t \in [0,1]} \int_0^1 G(t,s)g(s)ds \right]^{-1}, \text{ for } 0 \leq z \leq r;$$

then the boundary value problem (1) has triple symmetric positive solutions z_1, z_2, z_3 satisfy $\|z_1\| < d$, $m < u(z_2)$, and $\|z_3\| > d$ for $u(z_3) < m$, where

$$u(z) = \min_{t \in [\frac{1}{4}, \frac{3}{4}]} z(t).$$

Proof The proof is similar to Theorem 3.1, we omit it here.

Corollary 3.1 Suppose that (H'_1) and (H'_2) hold, and 3) in (H'_3) is replaced by the following condition

$$\lim_{z \rightarrow +\infty} \frac{h(z)}{z} < \left[\max_{t \in [0,1]} \int_0^1 G(t,s)g(s)ds \right]^{-1}.$$

Then the boundary value problem (1) has triple symmetric positive solutions.

4 Examples

Example 4.1 The following boundary value problem

$$\begin{cases} z''' + \frac{h(z)}{t(1-t)} = 0, & t \in [0, 1], \\ z(0) = z(1) = z'(0) = 0, \end{cases} \quad (2)$$

has triple symmetric positive solutions, where

$$h(z) = \begin{cases} 4z^2, & 0 \leq z < 2, \\ \frac{763}{2z-1}, & z \geq 2. \end{cases}$$

Proof Let

$$g(t) = \frac{1}{t(1-t)}.$$

Obviously $g(t)$ is singular at $t = 0$ and $t = 1$. $h(z)$ is continuous on $[0, +\infty)$. So condition (H'_1) holds.

Since

$$\int_0^1 G(\mu(s), s)g(s)ds = \frac{1}{2}(1 - \ln 2) < +\infty, \quad \int_{\frac{1}{4}}^{\frac{3}{4}} g(s)ds < +\infty,$$

then (H'_2) holds.

1) in (H'_3) follows immediately from

$$\begin{aligned} \left(\max_{t \in [0,1]} \int_0^1 G(t,s)g(s)ds \right)^{-1} &= \left(\int_0^1 G(\mu(s), s)g(s)ds \right)^{-1} = \frac{2}{1 - \ln 2}, \\ \left(\int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{4}, s\right)g(s)ds \right)^{-1} &= \left(\int_{\frac{1}{4}}^{\frac{3}{4}} \frac{1}{32}(1-s)^2 \frac{1}{s(1-s)}ds \right)^{-1} = \frac{32}{\ln 3 - 0.5}, \end{aligned}$$

we may take $d = \frac{1}{4}$, then

$$h(z) = 4z^2 \leq 4d^2 = \frac{1}{4} < \frac{2}{4(1 - \ln 2)} \quad \text{for } 0 \leq z \leq d = \frac{1}{4}.$$

2) in (H'_3) is immediate, since we may take $m = 2$, then

$$h(z) = \frac{763}{2z-1} \geq h(4) = 109 > 2 \times \frac{32}{\ln 3 - \frac{1}{2}}, \quad 2 \leq z \leq 4.$$

3) in (H'_3) is immediate since we may take $r = 100 > 2m = 4$, then

$$\max_{z \in [0,100]} h(z) \leq h(2) = \frac{763}{3} < 100 \times \frac{2}{1 - \ln 2}, \quad 0 \leq z \leq d = 100.$$

Thus, from Theorem 3.2, we know that the BVP (2) has triple symmetric positive solution z_1, z_2, z_3 satisfy $\|z_1\| < \frac{1}{4}$, $2 < u(z_2)$, and $\|z_3\| > \frac{1}{4}$ for $u(z_3) < 2$.

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三阶奇异边值问题对称正解的最优存在性

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摘 要: 本文利用 Leggett-Williams 不动点定理, 通过构造特殊的锥, 得到了三阶奇异边值问题三个对称正解最优存在性的新结果, 最后, 通过具体的例子说明了我们所得结果的有效性。

关键词: 对称正解; 存在性; 边值问题